Graph Fourier Transform Based on Directed Laplacian

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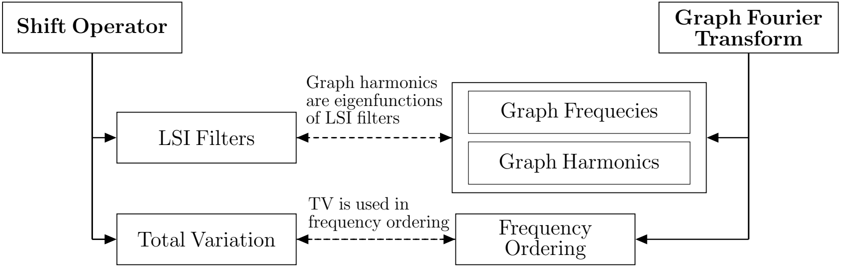
Abstract



Discrete Signal Processing on Graphs ($\mathrm{DSP_G}$) framework

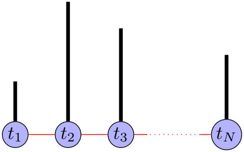
	Existing	**Proposed**
Shift Operator	Weight matrix	Derived from the directed Laplacian
Harmonics	Eigenvectors of the weight matrix	Eigenvectors of the directed Laplacian

- Achieved "natural" frequency ordering and interpretation

Outline

Background: Graph Signal Processing

- Classical signal processing



Discrete-time signal

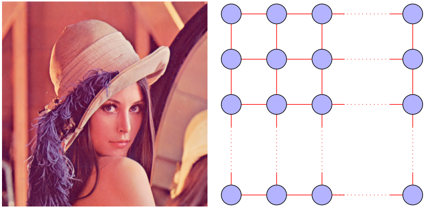
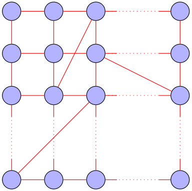
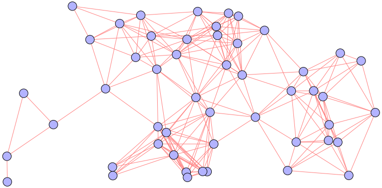


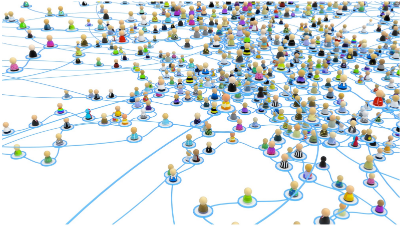
Image signal and its structure

- Graph signal processing

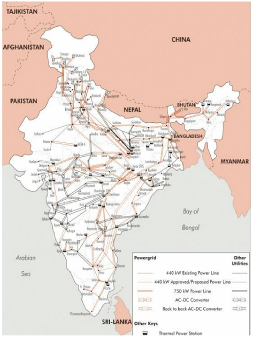




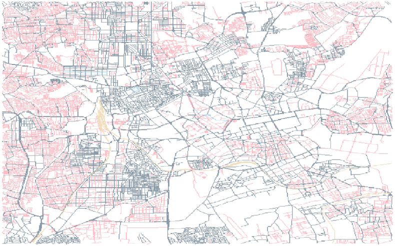
Graph Signal Processing Applications



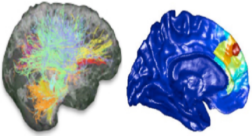
Social Network



Power Grid Network

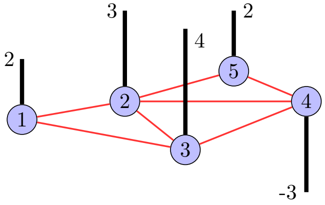


Bangalore Road Network



Biological Network

Notation



A graph signal **f**

Graph $\mathcal{G} = (\mathcal{V},\ \mathbf{W})$

$$\mathbf{f} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ -3 \\ 2 \end{bmatrix}$$

Weight matrix $\mathbf{W} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

Degree matrix $\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Laplacian matrix

$$\mathbf{L} = \mathbf{D} - \mathbf{W} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

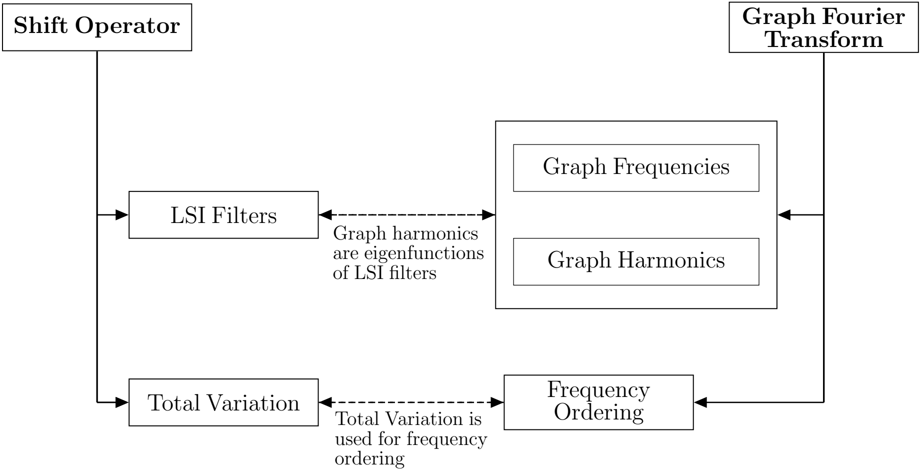
Existing Graph Signal Processing (GSP) Frameworks

	GSP based on Laplacian[1]	Discrete Signal Processing on Graphs (DSP_{G}) Framework[2]
Shift Operator	Not defined	The weight matrix
LSI Filters	Not applicable	Applicable
Applicability	Only undirected graphs	Directed graphs
Frequencies	Eigenvalues of the Laplacian matrix	Eigenvalues of the weight matrix
Harmonics	Eigenvectors of the Laplacian matrix	Eigenvectors of the weight matrix
Frequency Ordering	Laplacian quadratic form	Total variation
Multiscale Analysis	Exists (SGWT)	Does not exist

[1] David K Hammond, Pierre Vandergheynst, and Rémi Gribonval. "Wavelets on graphs via spectral graph theory". In: *Applied and Computational Harmonic Analysis* 30.2 (2011), pp. 129–150.

[2] A. Sandryhaila and J.M.F. Moura. "Discrete Signal Processing on Graphs: Frequency Analysis". In: *Signal Processing, IEEE Transactions on* 62.12 (June 2014), pp. 3042–3054.

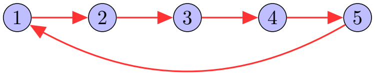
Discrete Signal Processing on Graphs (DSP_G) Framework



Concepts in the DSP_G framework

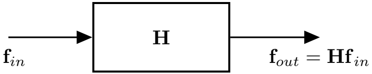
DSP_{G} Framework (Cont'd...)

- Shift operator
  - Weight matrix $\mathbf{W}$ of the graph
- Shifted graph signal $\boxed{\tilde{\mathbf{f}} = \mathbf{W}\mathbf{f}}$
- Example: shifting discrete-time signal (one unit right)



$$\mathbf{x} = [9,\ 7,\ 5,\ 0,\ 6]^T$$

$$\tilde{\mathbf{x}} = \mathbf{W}\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \\ 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 7 \\ 5 \\ 0 \end{bmatrix}$$



A linear graph filter

- Linear Shift Invariant (LSI) filters
  - $\mathbf{H}(\mathbf{W}\mathbf{f}_{in}) = \mathbf{W}(\mathbf{H}\mathbf{f}_{in})$
  - Polynomials in $\mathbf{W}$

$$\mathbf{H} = h(\mathbf{W}) = \sum_{m=0}^{M-1} h_m \mathbf{W}^m$$

$$= h_0 \mathbf{I} + h_1 \mathbf{W} + \ldots + h_{M-1} \mathbf{W}^{M-1}$$

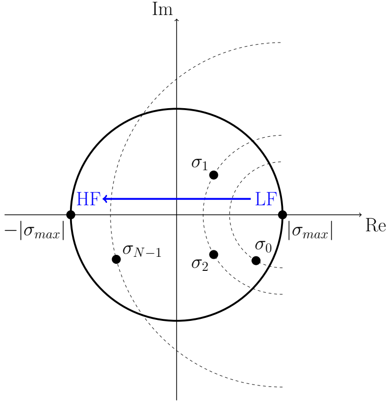
DSP_{G} Framework (Cont'd...)

- Analogy from classical signal processing
  - Classical Fourier basis: Complex exponentials
  - Complex exponentials are **Eigenfunctions** of Linear Time Invariant (LTI) filters
- Graph Fourier Transform
  - Graph Fourier basis are **Eigenfunctions** of Linear Shift Invariant (LSI) graph filters
  - Graph Frequencies: Eigenvalues of the weight matrix $\mathbf{W}$
  - Graph Harmonics: Eigenvectors of the weight matrix $\mathbf{W}$
  - $\boxed{\mathbf{W} = \mathbf{V}\boldsymbol{\Sigma}\mathbf{V}^{-1}}$
  - GFT $\boxed{\hat{\mathbf{f}} = \mathbf{V}^{-1}\mathbf{f}}$, IGFT $\boxed{\mathbf{f} = \mathbf{V}^{-1}\hat{\mathbf{f}}}$

DSP_G Framework (Cont'd...)

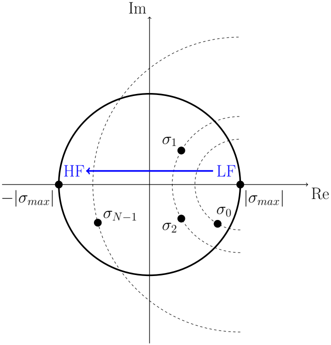
- Total Variation in classical signal processing
  - $\boxed{\mathrm{TV}(\mathbf{x}) = \sum_n |x[n] - x[n-1]| = ||\mathbf{x} - \tilde{\mathbf{x}}||_1}$, where $\tilde{x}[n] = x[n-1]$
- Analogy from classical signal processing
- Total Variation on graphs $\boxed{\mathrm{TV}_{\mathcal{G}}(\mathbf{f}) = ||\mathbf{f} - \tilde{\mathbf{f}}||_1 = ||\mathbf{f} - \mathbf{W}\mathbf{f}||_1}$

- Frequency ordering: Based on Total Variation
- Eigenvalue with largest magnitude: Lowest frquency

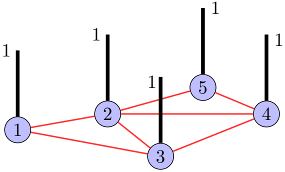


Motivation

Problems in Weight Matrix based $\mathrm{DSP_G}$



- Constant graph signal:
  High frequency components



$$\hat{\mathbf{f}} = \begin{bmatrix} 0 \\ \boxed{0.36} \\ \boxed{0.16} \\ 0 \\ 2.20 \end{bmatrix}$$

Graph frequencies:
-1.62, -1.47, -0.46, 0.62, 2.94

- Weight matrix based $\mathrm{DSP_G}$
  - Does not provide "natural" frequency ordering
  - Even a constant signal has high frequency components

Our Work

Graph Fourier Transform based on Directed Laplacian

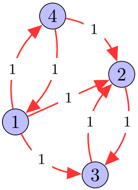
Graph Fourier Transform based on Directed Laplacian

- Redefined Graph Fourier Transform under $\mathrm{DSP_G}$
  - Shift operator: Derived from directed Laplacian
  - Linear Shift Invariant filters: Polynomials in the directed Laplacian
  - Graph frequencies: Eigenvalues of the directed Laplacian
  - Graph harmonics: Eigenvectors of the directed Laplacian
- "Natural" frequency ordering
- Better intuition of frequency as compared to the weight matrix based approach
- Coincides with the Laplacian based approach for undirected graphs

Directed Laplacian Matrix

- Basic matrices of a directed graph
  - Weight matrix: $\mathbf{W}$
    - w_{ij} is the weight of the directed edge from node j to node i
  - In-degree matrix: $\mathbf{D}_{\text{in}} = \text{diag}(\{d_i^{in}\}_{i=1,2,\ldots,N})$, $\quad d_i^{in} = \sum_{j=1}^{N} w_{ij}$
  - Out-degree matrix: $\mathbf{D}_{\text{out}} = \text{diag}(\{d_i^{out}\}_{i=1,2,\ldots,N})$, $\quad d_i^{out} = \sum_{i=1}^{N} w_{ij}$

- Directed Laplacian matrix $\boxed{\mathbf{L} = \mathbf{D}_{\text{in}} - \mathbf{W}}$
  - Sum of each row is zero
  - $\lambda = 0$ is surely an eigenvalue



A directed graph

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Weight matrix

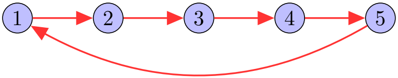
$$\mathbf{D}_{\text{in}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In-degree matrix

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Directed Laplacian matrix

Shift Operator (Proposed)



A directed cyclic (ring) graph

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

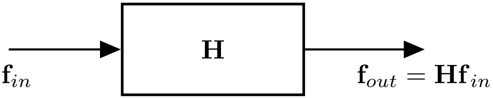
Laplacian matrix

- A signal $\mathbf{x} = [9\ 7\ 1\ 0\ 6]^T$; shifted by one unit to the right $\tilde{\mathbf{x}} = [6\ 9\ 7\ 1\ 0]^T$

$$\tilde{\mathbf{x}} = \mathrm{S}\mathbf{x} = (\mathrm{I} - \mathbf{L})\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \\ 1 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 7 \\ 1 \\ 0 \end{bmatrix}$$

- $\boxed{\mathrm{S} = (\mathrm{I} - \mathbf{L})}$ is the **shift operator**
- Shifted graph signal: $\boxed{\tilde{\mathbf{f}} = \mathrm{S}\mathbf{f} = (\mathrm{I} - \mathbf{L})\mathbf{f}}$

LSI Filters



A linear graph filter

- Linear Shift-Invariant (LSI) filter: $\boxed{\mathbf{S}(\mathbf{f}_{out}) = \mathbf{H}(\mathbf{S}\mathbf{f}_{in})}$

Theorem

A graph filter $\mathbf{H}$ *is LSI if the following conditions are satisfied.*

1. *Geometric multiplicity of each distinct eigenvalue of the graph Laplacian is one.*
2. *The graph filter* $\mathbf{H}$ *is a polynomial in* $\mathbf{L}$*, i.e., if* $\mathbf{H}$ *can be written as*

$$\mathbf{H} = h(\mathbf{L}) = h_0\mathbf{I} + h_1\mathbf{L} + \ldots + h_m\mathbf{L}^m$$

where, $h_0, h_1, \ \ldots \ , h_m \in \mathbb{C}$ *are called filter taps.*

Graph Fourier Transform based on Directed Laplacian

- Jordan decomposition of the directed Laplacian: $\boxed{\mathbf{L} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}}$
- **Graph Fourier basis**: Columns of $\mathbf{V}$ (Jordan Eigenvectors of $\mathbf{L}$)
- **Graph frequencies**: Eigenvalues of $\mathbf{L}$ (diagonal entries of Jordan blocks in $\mathbf{J}$)
- GFT $\boxed{\hat{\mathbf{f}} = \mathbf{V}^{-1}\mathbf{f}}$ and IGFT: $\boxed{\mathbf{f} = \mathbf{V}\hat{\mathbf{f}}}$
- **Frequency Ordering**: based on Total Variation

- Total Variation: $\mathrm{TV}_{\mathcal{G}}(\mathbf{f}) = ||\mathbf{f} - \mathbf{S}\mathbf{f}||_1 = ||\mathbf{f} - (\mathbf{I} - \mathbf{L})\mathbf{f}||_1$

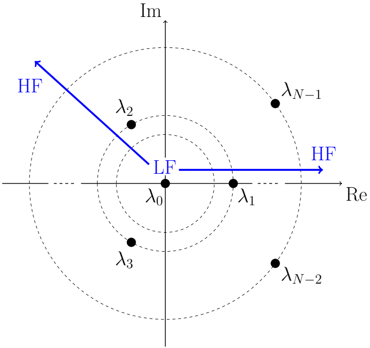
$$\boxed{\mathrm{TV}_{\mathcal{G}}(\mathbf{f}) = ||\mathbf{L}\mathbf{f}||_1}$$

Theorem

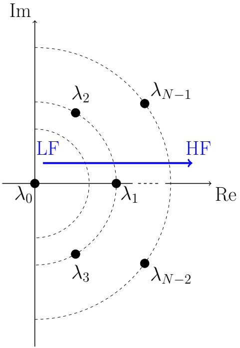
TV of an eigenvector $\mathbf{v}_r$ is proportional to the absolute value of the corresponding eigenvalue

$$\mathrm{TV}(\mathbf{v}_r) \propto |\lambda_r|$$

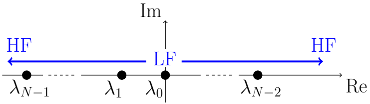
Frequency Ordering



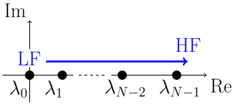
Arbitrary graph



Graph with positive edge weights



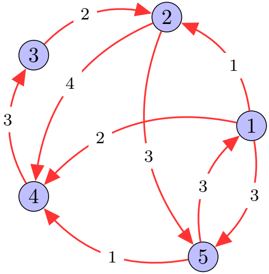
Undirected graph with real edge weights.



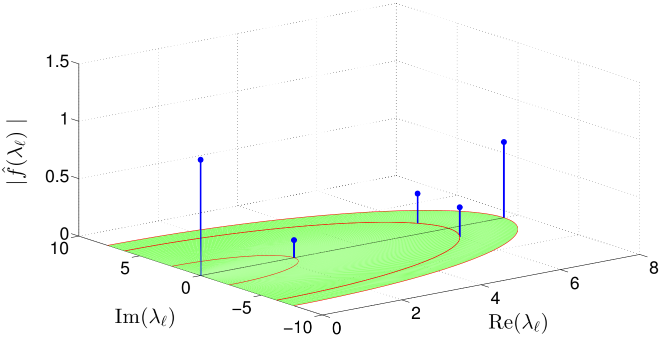
Undirected graph with real and non-negative edge weights.

Example

- Graph signal $\mathbf{f} = [0.1189\ 0.3801\ 0.8128\ 0.2441\ 0.8844]^T$ defined on the directed graph



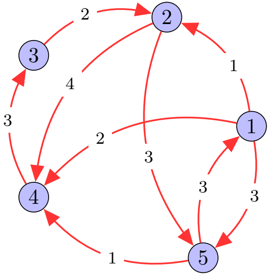
A weighted directed graph



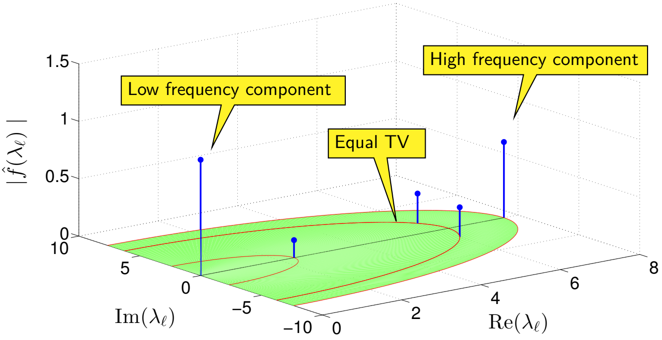
Spectrum of the signal $\mathbf{f} = [0.1189\ 0.3801\ 0.8128\ 0.2441\ 0.8844]^T$

Example

- Graph signal $\mathbf{f} = [0.1189\ 0.3801\ 0.8128\ 0.2441\ 0.8844]^T$ defined on the directed graph



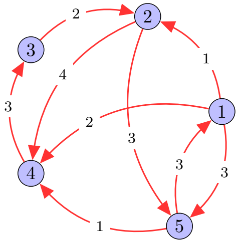
A weighted directed graph



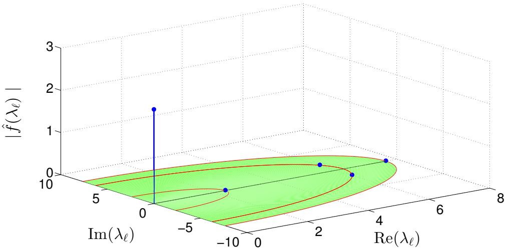
Spectrum of the signal $\mathbf{f} = [0.1189\ 0.3801\ 0.8128\ 0.2441\ 0.8844]^T$

Example: Zero Frequency

- Eigenvector corresponding to λ_0 is $\mathbf{v}_0 = \frac{1}{\sqrt{N}}[1,\ 1, \dots\ 1]^T$
  - TV of $\mathbf{v}_0$ is zero
- For a constant graph signal $\mathbf{f} = [k,\ k, \dots]^T$, GFT is $\hat{\mathbf{f}} = [(k\sqrt{N}),\ 0, \dots]^T$
  - Only zero frequency component



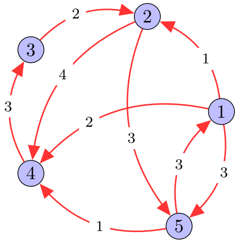
A weighted directed graph



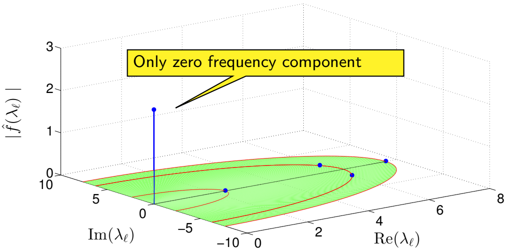
Spectrum of the constant signal $\mathbf{f} = [1\ 1\ 1\ 1\ 1]^T$

Example: Zero Frequency

- Eigenvector corresponding to λ_0 is $\mathbf{v}_0 = \frac{1}{\sqrt{N}}[1,\ 1, \dots\ 1]^T$
  - TV of $\mathbf{v}_0$ is zero
- For a constant graph signal $\mathbf{f} = [k,\ k, \dots]^T$, GFT is $\hat{\mathbf{f}} = [(k\sqrt{N}),\ 0, \dots]^T$
  - Only zero frequency component
- The **weight matrix based approach** of GFT **fails** to give this basic intuition



A weighted directed graph



Spectrum of the constant signal $\mathbf{f} = [1\ 1\ 1\ 1\ 1]^T$

Comparison of the GSP Frameworks

	GSP based on Laplacian	DSP_G Framework	
		Based on Weight Matrix	**Based on Directed Laplacian**
Shift Operator	Not defined	The weight matrix $\mathbf{W}$	Derived from directed Laplacian $(\mathbf{I} - \mathbf{L})$
LSI Filters	Not applicable	Applicable	Applicable
Applicability	Only undirected graphs	Directed graphs	Directed graphs
Frequencies	Eigenvalues of the Laplacian (real)	Eigenvalues of the weight matrix	Eigenvalues of the directed Laplacian
Harmonics	Eigenvectors of the Laplacian matrix (real)	Eigenvectors of the weight matrix	Eigenvectors of the directed Laplacian
Frequency Ordering	Laplacian quadratic form (natural)	Total variation (not natural)	Total variation (natural)
Multiscale Analysis	Exists (Spectral Graph Wavelet Transform)	Does not exist	Possible

Conclusions

- Redefined Graph Fourier Transform under the $\mathrm{DSP_G}$ framework
- Shift operator derived from directed Laplacian
- Eigendecomposition of directed Laplacian for frequency analysis
- "Natural" frequency ordering and interpretation
- Unification of existing approaches
- Spectral graph wavelet transform can be extended to directed graphs using directed Laplacian

References

[1] Fan RK Chung. *Spectral graph theory*. Vol. 92. American Mathematical Soc., 1997.

[2] David K Hammond, Pierre Vandergheynst, and Rémi Gribonval. "Wavelets on graphs via spectral graph theory". In: *Applied and Computational Harmonic Analysis* 30.2 (2011), pp. 129–150.

[3] Sunil K Narang and Antonio Ortega. "Perfect reconstruction two-channel wavelet filter banks for graph structured data". In: *Signal Processing, IEEE Transactions on* 60.6 (2012), pp. 2786–2799.

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[5] A. Sandryhaila and J.M.F. Moura. "Discrete Signal Processing on Graphs". In: *Signal Processing, IEEE Transactions on* 61.7 (Apr. 2013), pp. 1644–1656.

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[8] David I Shuman, Benjamin Ricaud, and Pierre Vandergheynst. "Vertex-frequency analysis on graphs". In: *Applied and Computational Harmonic Analysis* 40.2 (2016), pp. 260–291.

[9] David I Shuman et al. "The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains". In: *Signal Processing Magazine, IEEE* 30.3 (2013), pp. 83–98.

Thank You.

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